Fifteen puzzle.

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Last time

Definition

The permutation group S_n is the group of bijections of the set $\{1, 2, \ldots, n\}$.

It is convenient to denote permutations by

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & \ldots & n \\ \sigma(1) & \sigma(2) & \sigma(3) & \ldots & \sigma(n) \end{pmatrix}$$

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For $\sigma \in S_n$ define $inv(\sigma)$ to be the number of pairs (ij) such that $i < j$ but $\sigma(i) > \sigma(j)$. This number $inv(\sigma)$ is called the **number of inversions** of σ.

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Define the **sign** of σ to be $sgn(\sigma) = (-1)^{inv(\sigma)}$.

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- Prove that for any permutation σ, composing it with a transposition of neighbors $(i, i+1)$ either creates a new inversion, or removes one.

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- Thus for any representation of σ as a composition of N transpositions of neighbors, the sign $sgn(\sigma)$ is $(-1)^N$. (Need to be careful here.)
- Prove that for two permutations σ, τ we have $sgn(\sigma \circ \tau) = sgn(\sigma)sgn(\tau)$.

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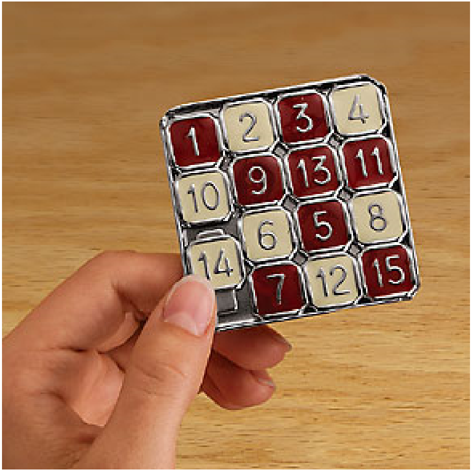
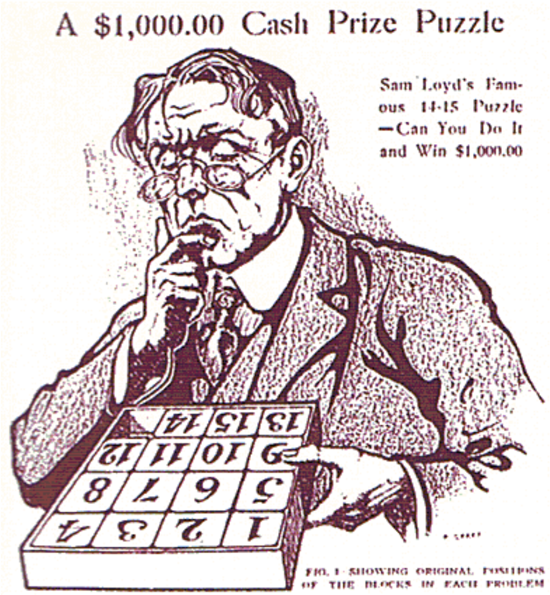
Let $A_n \subset S_n$ be the subset consisting of even permutations. A_n is called an **alternating GROUP** (check that it's a group!)

The Fifteen puzzle



Sam Loyd's puzzle



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- Prove that Sam Loyd's puzzle can't be solved.